Chapter 4

4-1 Let $x_1 = x$, $x_2 = y$, $u_1 = u$, and $u_2 = v$ in Eq. (4-17). Then

Equation, x - direction $$\frac{\partial P}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Equation, y - direction $$\frac{\partial P}{\partial y} = 0$$

Thus, $P = P(x)$ and $\frac{\partial P}{\partial x} = \frac{dP}{dx}$. The equation becomes

$$\frac{dP}{dx} = \mu \frac{d^2 u}{d y^2}$$

4-2 Consider the continuity equation (4-5) in cylindrical coordinates with $\nabla \cdot V$ defined in Appendix D,

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_x}{\partial x} = 0$$

where $V_r = v_r$; $V_\phi = v_\phi$; and $V_x = u$. Let $r = R + y$ and $x = R\phi$, where R is the radius of the inner surface. Note that $\partial/\partial x = 0$ (no axial flow). Thus

$$R\frac{\partial v_r}{\partial y} + y\frac{\partial v_r}{\partial y} + v_r + R\frac{\partial v_\phi}{\partial x} = 0$$

For v_r small and $y \ll R$, the middle two terms in the equation are negligible compared to the first and last terms. Thus the equation becomes identical to (4-7) in Cartesian coordinates. (Actually these are curvilinear coordinates with $y \ll R$). Using similar arguments, the applicable momentum equation is (4-11), with the convective acceleration terms neglected. The applicable energy equation is (4-38).

4-3 The appropriate set of coordinates is the fixed, non-rotating cylindrical system with r, the radial direction; ϕ, the circumferential direction; and x, the axial direction above the disk. In the governing equations, all three velocity components will appear, but derivatives with respect to ϕ will be zero due to rotational symmetry.

The flow over a rotating disk is boundary layer in character, but the complete Navier-Stokes equations can be solved in exact form (see Schlichting, ref. 2, page 93). The applicable energy equation is (4-35) with the conduction gradient in the r-direction neglected. (See Schlichting, page 296, for references to heat transfer solutions.)

4-4 The applicable energy equation is (4-31). Assume no mass diffusion. Use the definition of enthalpy, $i = e + P/\rho$, and let ρ be constant, yielding

$$\rho \frac{De}{Dt} - \nabla \cdot k \nabla T = S$$

For no fluid motion, the substantial derivative reduces to

$$\rho \frac{\partial e}{\partial t} - \nabla \cdot k \nabla T = S$$

This is the classic heat conduction equation for a solid where the thermal equation of state is $de = c dT$. For steady conduction and constant properties, the Poisson form of the conduction equation is obtained, and when S is equal to zero, the Laplace equation is obtained.

$$\nabla^2 T = \frac{S}{k}$$

4-5 Eq. (4-1) can be considered a continuity equation for the j - component of a mixture, but the creation term must be added, resulting in

$$\frac{\partial G_{tot,j,x}}{\partial x} + \frac{\partial G_{tot,j,y}}{\partial y} = \dot{m}_j'''$$

Now substitute $G_{tot,j} = G_{diff,j} + G_{conv,j}$, where $G_{conv,j} = m_j G$ and $G_{diff,j} = -\gamma_j \nabla m_j$. (Recall that G is the total mass flux vector.) Ignore $G_{diff,j,x}$ as a boundary layer approximation and substitute into the continuity equation.

$$m_j \left[\cancel{\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y}} \right] + G_x \frac{\partial m_j}{\partial x} + G_y \frac{\partial m_j}{\partial y} - \frac{\partial}{\partial y}\left(\gamma_j \frac{\partial m_j}{\partial y} \right) = m_j'''$$

4-6 For conservation of mass, the terms for mass flow rates (inflow on the radial face and the axial face) are

$$\dot{m}_r = G_r A_r \qquad G_r = \rho v_r; \quad A_r = r\,d\phi\,dx$$
$$\dot{m}_x = G_x A_x \qquad G_x = \rho u; \quad A_x = r\,d\phi\,dr$$

Application of the procedure leading up to equation (4-2), assuming steady flow, yields

$$\frac{\partial(u\rho)}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) = 0$$

Note the appearance of r comes from dividing through the equation by the differential volume, ($r\,d\phi\,dx$). For constant properties, Eq. (4-9) is found.

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0$$

For x-momentum , the x-momentum flow rates, inflow on the radial face and the axial face, and the x-forces are

$$\dot{M}_r = u\dot{m}_r = u\left(G_r A_r\right)$$
$$\dot{M}_x = u\dot{m}_x = u\left(G_x A_x\right)$$

and

$$F_x = -\sigma_x A_x - \tau_{rx} A_r \qquad \sigma_x \approx -P \qquad \tau_{rx} = \mu\frac{\partial u}{\partial y}$$

Application of the procedure leading up to Eqs. (4-10) and (4-11), assuming steady flow and constant properties yields

$$\rho u\frac{\partial u}{\partial x} + \rho v_r\frac{\partial u}{\partial r} = -\frac{dP}{dx} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

For energy , the energy flow rates, inflow on the radial face and on the axial face, and the corresponding heat and work rates are

$$\dot{E}_r = \left(i + \tfrac{1}{2}u^2\right)\dot{m}_r = \left(i + \tfrac{1}{2}u^2\right)G_r A_r$$
$$\dot{E}_x = \left(i + \tfrac{1}{2}u^2\right)\dot{m}_x = \left(i + \tfrac{1}{2}u^2\right)G_x A_x$$

and

$$\dot{q}_r = -kA_r\frac{\partial T}{\partial r} \qquad \dot{q}_x = -kA_x\frac{\partial T}{\partial x} \qquad \dot{W}_{shear,r} = u\tau_{rx}A_r$$

Application of the procedure leading up to Eqs. (4-27) and (4-28), assuming steady flow, no mass diffusion, and constant properties yields

$$\rho u \frac{\partial i}{\partial x} + \rho v_r \frac{\partial i}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}\left(kr \frac{\partial T}{\partial r} \right) - \mu \left(\frac{\partial u}{\partial r} \right)^2 - u \frac{dP}{dx} = 0$$

and for ideal gases and incompressible liquids, using the approximations similar to the development leading to Eq. (4-38), but retaining the pressure gradient term the enthalpy equation reduces to

$$u \frac{\partial T}{\partial x} + v_r \frac{\partial T}{\partial y} - \alpha \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \right] - \frac{\Pr}{c}\left(\frac{\partial u}{\partial r} \right)^2 - \frac{1}{\rho c}\frac{dP}{dx} = 0$$

4-7 These steps are basically the redevelopment of the formulation leading up to Eq. (4-39).